

COPPERSMITH-RIVLIN TYPE INEQUALITIES AND THE ORDER OF VANISHING OF POLYNOMIALS AT 1

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ABSTRACT. For $n \in \mathbb{N}$, $L > 0$, and $p \geq 1$ let $\kappa_p(n, L)$ be the largest possible value of k for which there is a polynomial $P \neq 0$ of the form

$$P(x) = \sum_{j=0}^n a_j x^j, \quad |a_0| \geq L \left(\sum_{j=1}^n |a_j|^p \right)^{1/p}, \quad a_j \in \mathbb{C},$$

such that $(x-1)^k$ divides $P(x)$. For $n \in \mathbb{N}$, $L > 0$, and $q \geq 1$ $\mu_q(n, L)$ be the smallest value of k for which there is a polynomial Q of degree k with complex coefficients such that

$$|Q(0)| > \frac{1}{L} \left(\sum_{j=1}^n |Q(j)|^q \right)^{1/q}.$$

We find the size of $\kappa_p(n, L)$ and $\mu_q(n, L)$ for all $n \in \mathbb{N}$, $L > 0$, and $1 \leq p, q \leq \infty$. The result about $\mu_\infty(n, L)$ is due to Coppersmith and Rivlin, but our proof is completely different and much shorter even in that special case.

1. NOTATION

In [B-99] and [B-13] we examined a number of problems concerning polynomials with coefficients restricted in various ways. We are particularly interested in how small such polynomials can be on the interval $[0, 1]$. For example, we proved that there are absolute constants $c_1 > 0$ and $c_2 > 0$ such that

$$\exp(-c_1 \sqrt{n}) \leq \min_{0 \neq Q \in \mathcal{F}_n} \left\{ \max_{x \in [0, 1]} |Q(x)| \right\} \leq \exp(-c_2 \sqrt{n})$$

for every $n \geq 2$, where \mathcal{F}_n denotes the set of all polynomials of degree at most n with coefficients from $\{-1, 0, 1\}$.

Littlewood considered minimization problems of this variety on the unit disk. His most famous, now solved, conjecture was that the L_1 norm of an element $f \in \mathcal{F}_n$ on the unit circle grows at least as fast as $c \log N$, where N is the number of non-zero coefficients in f and $c > 0$ is an absolute constant.

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When the coefficients are required to be integers, the questions have a Diophantine nature and have been studied from a variety of points of view. See [A-79, B-98, B-95, F-80, O-93].

One key to the analysis is a study of the related problem of giving an upper bound for the multiplicity of the zero these restricted polynomials can have at 1. In [B-99] and [B-13] we answer this latter question precisely for the class of polynomials of the form

$$Q(x) = \sum_{j=0}^n a_j x^j, \quad |a_j| \leq 1, \quad a_j \in \mathbb{C}, \quad j = 1, 2, \dots, n,$$

with fixed $|a_0| \neq 0$.

Variants of these questions have attracted considerable study, though rarely have precise answers been possible to give. See in particular [A-90, B-32, B-87, E-50, Sch-33, Sz-34]. Indeed, the classical, much studied, and presumably very difficult problem of Prouhet, Tarry, and Escott rephrases as a question of this variety. (Precisely: what is the maximal vanishing at 1 of a polynomial with integer coefficients with l_1 norm $2n$? It is conjectured to be n .) See [H-82], [B-94], or [B-02].

For $n \in \mathbb{N}$, $L > 0$, and $p \geq 1$ we define the following numbers. Let $\kappa_p(n, L)$ be the largest possible value of k for which there is a polynomial $Q \neq 0$ of the form

$$Q(x) = \sum_{j=0}^n a_j x^j, \quad |a_0| \geq L \left(\sum_{j=1}^n |a_j|^p \right)^{1/p}, \quad a_j \in \mathbb{C},$$

such that $(x-1)^k$ divides $Q(x)$. For $n \in \mathbb{N}$ and $L > 0$ let $\kappa_\infty(n, L)$ the largest possible value of k for which there is a polynomial $Q \neq 0$ of the form

$$Q(x) = \sum_{j=0}^n a_j x^j, \quad |a_0| \geq L \max_{1 \leq j \leq n} |a_j|, \quad a_j \in \mathbb{C},$$

such that $(x-1)^k$ divides $Q(x)$. In [B-13] we proved that there is an absolute constant $c_3 > 0$ such that

$$\min \left\{ \frac{1}{6} \sqrt{n(1 - \log L)} - 1, n \right\} \leq \kappa_\infty(n, L) \leq \min \left\{ c_3 \sqrt{n(1 - \log L)}, n \right\}$$

for every $n \in \mathbb{N}$ and $L \in (0, 1]$. However, we were far from being able to establish the right result in the case of $L \geq 1$. In [B-13] we proved the right order of magnitude of $\kappa_\infty(n, L)$ and $\kappa_2(n, L)$ in the case of $L \geq 1$. Our results in [B-99] and [B-13] sharpen and generalize results of Schur [Sch-33], Amoroso [A-90], Bombieri and Vaaler [B-87], and Hua [H-82] who gave versions of this result for polynomials with integer coefficients. Our results in [B-99] have turned out to be related to a number of recent papers from a rather wide range of research areas. See [A-02, B-98, B-95, B-96 B-97a, B-97b, B-97, B-00, B-07, B-08a, B-08b, Bu-99, C-02, C-13, C-10, D-99, D-01, D-03, D-13, E-08a, E-08b, F-00, G-05, K-04, K-09, M-03, M-68, N-94, O-93, P-99, P-12, P-13, R-04, R-07, S-99, T-07, T-84], for example.

More on the zeros of polynomials with Littlewood-type coefficient constraints may be found in [E-02b]. Markov and Bernstein type inequalities under Erdős type coefficient constraints are surveyed in [E-02a].

For $n \in \mathbb{N}$, $L > 0$, and $q \geq 1$ we define the following numbers. Let $\mu_q(n, L)$ be the smallest value of k for which there is a polynomial of degree k with complex coefficients such that

$$|Q(0)| > \frac{1}{L} \left(\sum_{j=1}^n |Q(j)|^q \right)^{1/q}.$$

Let $\mu_\infty(n, L)$ be the smallest value of k for which there is a polynomial of degree k with complex coefficients such that

$$|Q(0)| > \frac{1}{L} \max_{j \in \{1, 2, \dots, n\}} |Q(j)|.$$

It is a simple consequence of Hölder's inequality (see Lemma 3.6) that

$$\kappa_p(n, L) \leq \mu_q(n, L).$$

whenever $n \in \mathbb{N}$, $L > 0$, $1 \leq p, q \leq \infty$, and $1/p + 1/q = 1$.

In this paper we find the size of $\kappa_p(n, L)$ and $\mu_q(n, L)$ for all $n \in \mathbb{N}$, $L > 0$, and $1 \leq p, q \leq \infty$. The result about $\mu_\infty(n, L)$ is due to Coppersmith and Rivlin, [C-92], but our proof presented in this paper is completely different and much shorter even in that special case.

2 NEW RESULTS

Theorem 2.1. *Let $p \in (1, \infty]$ and $q \in [1, \infty)$ satisfy $1/p + 1/q = 1$. There are absolute constants $c_1 > 0$ and $c_2 > 0$ such that*

$$\sqrt{n}(c_1 L)^{-q/2} - 1 \leq \kappa_p(n, L) \leq \mu_q(n, L) \leq \sqrt{n}(c_2 L)^{-q/2} + 2$$

for every $n \in \mathbb{N}$ and $L > 1/2$, and

$$c_3 \min \left\{ \sqrt{n(-\log L)}, n \right\} \leq \kappa_p(n, L) \leq \mu_q(n, L) \leq c_4 \min \left\{ \sqrt{n(-\log L)}, n \right\} + 4$$

for every $n \in \mathbb{N}$ and $L \in (0, 1/2]$. Here $c_1 := 1/53$, $c_2 := 40$, $c_3 := 2/7$, and $c_4 := 13$ are appropriate choices.

Theorem 2.2. *There are constants absolute $c_1 > 0$ and $c_2 > 0$ such that*

$$c_1 \sqrt{n(1-L)} - 1 \leq \kappa_1(n, L) \leq \mu_\infty(n, L) \leq c_2 \sqrt{n(1-L)} + 1$$

for every $n \in \mathbb{N}$ and $L \in (1/2, 1]$, and

$$c_3 \min \left\{ \sqrt{n(-\log L)}, n \right\} \leq \kappa_1(n, L) \leq \mu_\infty(n, L) \leq c_4 \min \left\{ \sqrt{n(-\log L)}, n \right\} + 4$$

for every $n \in \mathbb{N}$ and $L \in (0, 1/2]$. Note that $\kappa_1(n, L) = \mu_\infty(n, L) = 0$ for every $n \in \mathbb{N}$ and $L > 1$. Here $c_1 := 1/5$, $c_2 := 1$, $c_3 := 2/7$, and $c_4 := 13$ are appropriate choices.

3. LEMMAS

In this section we list our lemmas needed in the proofs of Theorems 2.1 and 2.2. These lemmas are proved in Section 4. Let \mathcal{P}_n be the set of all polynomials of degree at most n with real coefficients. Let \mathcal{P}_n^c be the set of all polynomials of degree at most n with complex coefficients.

Lemma 3.1. *Let $p \in (1, \infty)$. For any $1 \leq M$ there are polynomials P_n of the form*

$$P_n(x) = \sum_{j=0}^n a_{j,n} x^j, \quad a_{j,n} \in \mathbb{R}, \quad a_{0,n} \geq \frac{3M}{\pi^2} + o(M),$$

$$\left(\sum_{j=1}^n |a_{j,n}|^p \right)^{1/p} \leq 16M^{1/p},$$

such that P_n has at least $\lfloor \sqrt{n/M} \rfloor$ zeros at 1.

Lemma 3.2. *Let $p, q \in (1, \infty)$ satisfy $1/p + 1/q = 1$. For any $L \geq 1/48$ there are polynomials P_n of the form*

$$P_n(x) = \sum_{j=0}^n a_{j,n} x^j, \quad a_{j,n} \in \mathbb{R}, \quad a_{0,n} \geq L + o(L), \quad \sum_{j=1}^n |a_{j,n}|^p \leq 1,$$

such that P_n has at least $\lfloor \sqrt{n}(cL)^{-q/2} \rfloor$ zeros at 1 with $c := \frac{3}{16\pi^2}$.

Lemma 3.3. *Let $p \in [1, \infty)$. For any $L \in (0, 1/17)$ there are polynomials P_n of the form*

$$P_n(x) = \sum_{j=0}^n a_{j,n} x^j, \quad a_{j,n} \in \mathbb{R}, \quad a_{0,n} = L, \quad \sum_{j=1}^n |a_{j,n}|^p \leq 1,$$

such that P_n has at least $\frac{2}{7} \min\{\sqrt{n(1 - \log L)}, n\}$ zeros at 1.

Lemma 3.4. *For any $L \in (0, 1)$ there are polynomials $P_n \not\equiv 0$ of the form*

$$P_n(x) = \sum_{j=0}^n a_{j,n} x^j, \quad a_{j,n} \in \mathbb{R}, \quad a_{0,n} \geq L \sum_{j=1}^n |a_{j,n}|,$$

such that P_n has at least $\frac{1}{5} \sqrt{(n-1)(1-L)}$ zeros at 1.

The observation below is well known, easy to prove, and recorded in several papers. See [B-99], for example.

Lemma 3.5. *Let $P \neq 0$ be a polynomial of the form $P(x) = \sum_{j=0}^n a_j x^j$. Then $(x-1)^k$ divides P if and only if $\sum_{j=0}^n a_j Q(j) = 0$ for all polynomials $Q \in \mathcal{P}_{k-1}^c$.*

Our next lemma is a simple consequence of Hölder's inequality.

Lemma 3.6. *Let $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$. Then for every $n \in \mathbb{N}$ and $L > 0$, we have*

$$\kappa_p(n, L) \leq \mu_q(n, L).$$

The next lemma is stated as Lemma 3.4 in [K-03], where a proof of it is also presented.

Lemma 3.7. *For arbitrary real numbers $A, M > 0$, there exists a polynomial G such that $F = G^2 \in \mathcal{P}_m$ with*

$$m < \sqrt{\pi} \sqrt{A} \sqrt[4]{M} + 2$$

such that $F(0) = M$ and

$$|F(x)| \leq \min\{M, x^{-2}\}, \quad x \in (0, A].$$

We also need Lemma 5.7 from [B-99] which may be stated as follows.

Lemma 3.8. *Let n and R be positive integers with $1 \leq R \leq \sqrt{n}$. Then there exists a polynomial $F \in \mathcal{P}_m$ with*

$$m \leq 4\sqrt{n} + \frac{9}{7}R\sqrt{n} + R + 4 \leq \frac{44}{7}R\sqrt{n} + 4$$

such that

$$F(1) = F(2) = \cdots = F(R^2) = 0$$

and

$$|F(0)| > \exp(R^2)(|F(R^2 + 1)| + |F(R^2 + 2)| + \cdots + |F(n)|) \geq \exp(R^2) \left(\sum_{j=1}^n |F(j)|^2 \right)^{1/2}.$$

Lemmas 3.6 and 3.7 imply the following results needed in the proof of Theorems 2.1 and 2.2.

Lemma 3.9. *Let $q \in [1, \infty)$. For every $n \in \mathbb{N}$, $q \in [1, \infty)$, and $K > 0$, there are polynomials $F \in \mathcal{P}_m$ satisfying*

$$|F(0)| > K \left(\sum_{j=1}^n |F(j)|^q \right)^{1/q} \quad \text{and} \quad m \leq \begin{cases} \sqrt{n}(40K)^{q/2} + 2, & 0 < K < 2, \\ 13 \min \left\{ \sqrt{n \log K}, n \right\} + 4, & K \geq 2. \end{cases}$$

Lemma 3.10. *For every $n \in \mathbb{N}$ and $K > 1$, there are polynomials $F \in \mathcal{P}_m$ satisfying*

$$|F(0)| > K \max_{j \in \{1, 2, \dots, n\}} |F(j)|, \quad \text{and} \quad m \leq \begin{cases} \sqrt{n(K-1)/2} + 1, & 1 < K < 2, \\ 13 \min \left\{ \sqrt{n \log K}, n \right\} + 4, & K \geq 2. \end{cases}$$

4. PROOFS OF THE LEMMAS

Proof of Lemma 3.1. Modifying the construction on page 138 of [B-95] we define $H_1(x) := 1$ and

$$H_m(x) := \frac{(-1)^{m+1} 2(m!)^2}{2\pi i} \int_{\Gamma} \frac{x^t dt}{(t-2) \prod_{j=0}^m (t-j^2)}, \quad m = 2, 3, \dots, \quad x \in (0, \infty),$$

where the simple closed contour Γ surrounds the zeros of the denominator of the integrand. Then H_m is a polynomial of degree m^2 with a zero at 1 with multiplicity at least $m+1$. (This can be seen easily by repeated differentiation and then evaluation of the above contour integral by expanding the contour to infinity.) Also, by the residue theorem,

$$(4.1) \quad H_m(x) = 1 + d_m x^2 + \sum_{k=1}^m c_{k,m} x^{k^2}, \quad m = 2, 3, \dots,$$

where

$$c_{k,m} = \frac{(-1)^{m+1} 2(m!)^2}{(k^2-2) \prod_{j=0, j \neq k}^m (k^2-j^2)} = \frac{4}{k^2-2} \frac{(-1)^{k+1} (m!)^2}{(m-k)!(m+k)!},$$

and

$$d_m = \frac{(-1)^{m+1} 2(m!)^2}{\prod_{j=0}^m (2-j^2)}.$$

It follows that each $c_{k,m}$ is real and

$$(4.2) \quad |c_{k,m}| \leq \frac{4}{|k^2-2|}, \quad k = 1, 2, \dots, m,$$

and a simple calculation shows that

$$(4.3) \quad |d_m| \leq 8, \quad m = 2, 3, \dots$$

(No effort has been made to optimize the bound in (4.3).) Let S_M be the collection of all odd square free integers in $[1, M]$. Let $m := \lfloor \sqrt{n/M} \rfloor$. If $m = 0$ then there is nothing to prove. So we may assume that $m \geq 1$. It is well known that

$$|S_M| \geq \frac{3M}{\pi^2} + o(M),$$

where $|A|$ denotes the number of elements in a finite set a . This follows from the fact that if S_M^* is the collection of all square free integers in $[1, M]$, then

$$|S_M^*| = \frac{6M}{\pi^2} + o(M),$$

see [H-38, pp. 267-268], for example, by observing that the number of odd square free integer in $[1, M]$ is not less than the number of even square free integers in $[1, M]$ (if a is an even square free integer then $a/2$ is an odd square free integer). We define

$$P_n(x) := \sum_{j \in S_M} H_m(x^j).$$

Then P_n is of the form

$$P_n(x) = \sum_{j=0}^n a_{j,n} x^j, \quad a_{j,n} \in \mathbb{R}, \quad j = 0, 1, \dots, n.$$

We have

$$a_{0,n} = |S_M^*| \geq \frac{3M}{\pi^2} + o(M).$$

First assume that $m = 1$. Then

$$\sum_{j=1}^n |a_{j,n}|^p = 2|S_M| \leq 2M,$$

and as P_n has 1 zero at 1, the lemma follows. Now assume that $m \geq 2$. Since $ju \neq lv$ whenever $j, l \in S_M$, $j \neq l$, and $u, v \in \{1^2, 2^2, \dots, m^2\} \cup \{2\}$, we have

$$\begin{aligned} \sum_{j=1}^n |a_{j,n}|^p &\leq |S_M| \left(8^p + \sum_{k=1}^m \left(\frac{4}{|k^2 - 2|} \right)^p \right) \leq |S_M| \left(8^p + \sum_{k=1}^m \frac{4^p}{|k^2 - 2|} \right) \\ &= M(8^p + 8^p) \leq 16^p M. \end{aligned}$$

Observe that each term in P_n has a zero at 1 with multiplicity at least $m + 1 > \lfloor \sqrt{n/M} \rfloor$ zeros at 1, and hence so does P_n . \square

Proof of Lemma 3.2. The statement follows from Lemma 3.1 by choosing $1 \leq M$ so that

$$L := \frac{3}{16\pi^2} M^{1-1/p} = \frac{3}{16\pi^2} M^{1/q}.$$

This can be done when $\frac{3}{16\pi^2} \leq L$. \square

Proof of Lemma 3.3. Let $L \in (0, 1/17]$. We define

$$k := \min \left\{ \left\lfloor \frac{-\log L}{\log 17} \right\rfloor, n \right\} \quad \text{and} \quad m := \lfloor \sqrt{n/k} \rfloor.$$

Observe that $k \geq 1$ and $m \geq 1$ hold. Let $P_n := LH_m^k \in \mathcal{P}_n$, where $H_m \in \mathcal{P}_{m^2}$ defined by (4.1). Then

$$P_n(x) = \sum_{j=0}^n a_{j,n} x^j, \quad a_{j,n} \in \mathbb{R}, \quad j = 0, 1, \dots, n,$$

has at least

$$km \geq k \frac{1}{2} \sqrt{n/k} = \frac{1}{2} \sqrt{nk} = \frac{1}{2\sqrt{\log 17}} \min \left\{ \sqrt{n(-\log L)}, n \right\}$$

zeros at 1, where $2\sqrt{\log 17} < 7/2$. Clearly, $a_{0,n} = P_n(0) = L$, and using the notation in (4.1), we can deduce that

$$\begin{aligned} \sum_{j=1}^n |a_{j,n}|^p &\leq L^p \left(\sum_{j=1}^n |a_{j,n}| \right)^p \leq L^p \left(1 + |d_m| + \sum_{k=1}^m |c_{k,m}| \right)^{kp} \\ &\leq L^p (1 + 8 + 8)^{kp} = L^p 17^{kp} \leq L^p L^{-p} = 1, \end{aligned}$$

if $m \geq 2$, and

$$\sum_{j=1}^n |a_{j,n}|^p \leq L^p \left(\sum_{j=1}^n |a_{j,n}| \right)^p \leq L^p 2^{kp} \leq L^p L^{-p} = 1,$$

if $m = 1$. \square

Proof of Lemma 3.4. Let

$$r := \left\lfloor 12 \frac{1+L}{1-L} \right\rfloor + 1 \quad \text{and} \quad m := \left\lfloor \sqrt{\frac{n-1}{r}} \right\rfloor.$$

When $m \leq 1$ we have $\lfloor (1/9)\sqrt{n(1-L)} \rfloor = 0$, so there is nothing to prove. Now assume that $m \geq 2$. Let $P_n \in \mathcal{P}_n$ be defined by $P_n(x) := H_m(x^r)$, where $H_m \in \mathcal{P}_{m^2}$ defined by (4.1). Let $Q_n \in \mathcal{P}_n$ be defined by

$$Q_n(x) = - \int_0^1 P_n(t) dt + \int_0^x P_n(t) dt.$$

Then, using the notation in (4.1), we have

$$Q_n(x) = -1 - \frac{d_m}{2r+1} - \sum_{k=1}^m \frac{c_{k,m}}{rk^2+1} + x + \frac{d_m x^{2r+1}}{2r+1} + \sum_{k=1}^m \frac{c_{k,m} x^{rk^2+1}}{rk^2+1}.$$

Writing

$$Q_n(x) = \sum_{j=0}^n a_{j,n} x^j, \quad a_{j,n} \in \mathbb{R}, \quad j = 0, 1, \dots, n,$$

and recalling (4.2) and (4.3), we have

$$|a_{0,n}| \geq 1 - \frac{8}{2r+1} - \sum_{k=1}^m \frac{4}{|k^2-4|(rk^2+1)} \geq 1 - \frac{8}{2r+1} - \frac{8}{r} > 1 - \frac{12}{r},$$

and

$$\sum_{j=1}^n |a_{j,n}| \leq 1 + \frac{8}{2r+1} + \sum_{k=1}^m \frac{4}{(k^2-2)(rk^2+1)} < 1 + \frac{12}{r}.$$

Combining the previous two inequalities, we obtain

$$\frac{|a_{0,n}|}{\sum_{j=1}^n |a_{j,n}|} > \frac{1-12/r}{1+12/r} \geq \frac{1-(1-L)/(1+L)}{1+(1-L)/(1+L)} = L.$$

Also Q_n has at least $m+1 \geq \lfloor \sqrt{(n-1)/r} \rfloor + 1 \geq \frac{1}{5} \sqrt{(n-1)(1-L)}$ zeros at 1. \square

Proof of Lemma 3.6. We assume that $p, q \in (1, \infty)$, the result in the cases $p=1, q=\infty$ and $p=\infty, q=1$ can be proved similarly with straightforward modification of the proof. Let $m := \mu_q(n, L)$. Let Q be a polynomial of degree m with complex coefficients such that

$$|Q(0)| > \frac{1}{L} \left(\sum_{j=1}^n |Q(j)|^q \right)^{1/q}.$$

Now let P be a polynomial of the form

$$P(x) = \sum_{j=0}^n a_j x^j, \quad |a_0| \geq L \left(\sum_{j=1}^n |a_j|^p \right)^{1/p}, \quad a_j \in \mathbb{C}.$$

It follows from Hölder's inequality that

$$\left| \sum_{j=1}^n a_j Q(j) \right| \leq \left(\sum_{j=1}^n |a_j|^p \right)^{1/p} \left(\sum_{j=1}^n |Q(j)|^q \right)^{1/q} < \frac{|a_0|}{L} L |Q(0)| = |a_0 Q(0)|.$$

Then $\sum_{j=0}^n a_j Q(j) \neq 0$, and hence Lemma 3.5 implies that $(x-1)^{m+1}$ does not divide P . We conclude that $\kappa_p(n, L) \leq m = \mu_q(n, L)$. \square

Proof of Lemma 3.9. Note that $\mu_q(n, K) \leq n$ for all $n \in \mathbb{N}$ and $L > 0$, as it is shown by $H \in \mathcal{P}_n$ defined by $H(x) := \prod_{j=1}^n (x-j)$.

Case 1: $0 < K < n^{-1/q}$. The choice $F \equiv 1$ gives the lemma.

Case 2: $n^{-1/q} \leq K < 2$. Let F be the polynomial given in Lemma 3.7 with $A := n$ and $M := (4K)^{2q}$. Then

$$\begin{aligned} \sum_{j=1}^n |F(j)|^q &\leq \sum_{j \leq M^{-1/2}} M^q + \sum_{j > M^{1/2}} \frac{1}{j^{2q}} < M^{q-1/2} + \frac{1}{2q-1} [M^{-1/2}]^{-2q+1} \\ &\leq (1 + 2^{2q-1}) M^{q-1/2}, \end{aligned}$$

so

$$\left(\sum_{j=1}^n |F(j)|^q \right)^{1/q} < 4M^{1-1/(2q)} = K^{-1}F(0),$$

and the degree m of F satisfies

$$m < \pi\sqrt{n}\sqrt[4]{M} + 2 < \pi\sqrt{n}(4K)^{q/2} + 2 \leq \sqrt{n}(40K)^{q/2} + 2.$$

Case 3: $2 \leq K \leq \exp(n - 2\sqrt{n})$. Let $R := \lfloor \sqrt{\log K} \rfloor + 1$, and let F be the polynomial given in Lemma 3.7 with this R . Then

$$|F(0)| > K \sum_{j=1}^n |F(j)| \geq K \left(\sum_{j=1}^n |F(j)|^q \right)^{1/q},$$

and the degree m of F satisfies

$$m \leq \frac{44}{7}R\sqrt{n} + 4 \leq 13\sqrt{n \log K} + 4.$$

Case 4: $K > \exp(n - 2\sqrt{n})$, $n \geq 9$. Then $\log K > n - 2\sqrt{n} \geq n/3$ for all $n \geq 9$. Hence the polynomial $F \in \mathcal{P}_n$ defined by $F(x) := \prod_{j=1}^n (x - j)$ shows that

$$\mu_q(n, K) \leq n \leq \sqrt{3} \min \left\{ \sqrt{n \log K}, n \right\}.$$

Case 5: $K \geq 2$ and $n < 9$. Now the polynomial $F \in \mathcal{P}_n$ defined by $F(x) := \prod_{j=1}^n (x - j)$ shows

$$\mu_q(n, K) \leq n \leq 4 \min \left\{ \sqrt{n \log K}, n \right\}.$$

□

Proof of Lemma 3.10. First let $1 < K < 2$. Let $m = \lfloor \sqrt{n(K-1)/2} \rfloor + 1$. Let T_m be the Chebyshev polynomial of degree m defined by

$$T_m(\cos t) = \cos(mt), \quad t \in \mathbb{R}.$$

It is well known that $|T'_m(1)| = m^2$ and $T'_m(x)$ is increasing on $[1, \infty)$, hence $T_m(1+x) \geq 1 + m^2x$ for all $x > 0$. Now we define $F \in \mathcal{P}_m$ by

$$F(x) := T_m \left(\frac{-2x}{n-1} + \frac{n+1}{n-1} \right).$$

Then $|F(x)| \leq 1$ for all $x \in [1, n]$, and

$$F(0) \geq T_m \left(1 + \frac{2}{n-1} \right) > 1 + \frac{m^2}{n-1} > 1 + \frac{m^2}{n} \geq K,$$

which finishes the proof in the case of $1 < K < 2$. Now let $k \geq 2$. Then the polynomial $F \in \mathcal{P}_m$ chosen for $q = 1$, $n \in \mathbb{N}$, and $K \geq 2$ by Lemma 3.9 gives that

$$|F(0)| > K \left(\sum_{j=1}^n |F(j)|^q \right)^{1/q} \geq K \max_{j \in \{1, 2, \dots, n\}} |F(j)|,$$

with

$$m \leq 13 \min \left\{ \sqrt{n \log K}, n \right\} + 4.$$

□

5. PROOFS OF THE THEOREMS

Proof of Theorem 2.1. Without loss of generality we may assume that $p \in (1, \infty)$, as the case $p = \infty$ follows by a simple limiting argument (or we may as well refer to the main result in [B-13]). By Lemma 3.6 we have

$$\kappa_p(n, L) \leq \mu_q(n, L)$$

for every $n \in \mathbb{N}$ and $L > 0$. The lower bounds for $\kappa_p(n, L)$ follows from Lemmas 3.2 and 3.3. The upper bounds for $\mu_q(n, L)$ follow from Lemma 3.9 with $K = L^{-1}$. \square

Proof of Theorem 2.2. By Lemma 3.6 we have

$$\kappa_1(n, L) \leq \mu_\infty(n, L)$$

for every $n \in \mathbb{N}$ and $L > 0$. The lower bounds for $\kappa_1(n, L)$ follow from Lemmas 3.3 and 3.4. The upper bounds for $\mu_\infty(n, L)$ follow from Lemma 3.10 with $K = L^{-1}$. \square

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